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MEMORANDUM

13 February 1975

To: C. H. Kaman
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From: W. G. Flannelly

Subj: Modal Energies in a Guitar Structure

Because the problem apparently comes down to principally one of management of the amount of vibratory energy in the various modes of a guitar structure, considering that the problems of tuning and damping are better understood, the first approach was to find a way to measure the player-independent parameters controlling the entire kinetic energy of the guitar in each of its natural modes. The attached Working Paper shows that this narrows down to one parameter, the modal accelerations at the bridge, and to only one measurement over the spectrum if it is reasonable to approximate the input forces from the strings as combining into a single shear force. The Working Paper presents some conclusions on why some modes have high energy content and some low and gives a few examples of how these can be changed.

The derivation down to use of a single shear force is mathematically rigorous, barring a mistake, under the assumptions of linearality and that the structure possesses normal modes. These assumptions are not correct for a guitar but they are necessary for the construction of thinking tools to help determine trends. People who concentrate on non-linear analyses, usually professors, almost always assume that the particular nonlinear element is the only one in the structure that exists and that is not very useful. von Karman once commented that dynamicists "are capable of assuming anything except responsibility".

The finding shown in Equation (4), which is new insofar as I know, will be equally as useful in our contract research as it may be to guitars and the proportion of company time spent on it seems justifiably charge-able to Engineering Overhead if there is no objection.

Bill

W. G. Flannelly

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Attachment

MODAL COMPONENTS IN THE KINETIC ENERGY OF A GUITAR
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At any given forcing frequency, the kinetic energy is, by definition,

$$KE = \frac{1}{2}\{\dot{y}\}^T[M]\{\bar{\dot{y}}\} \tag{1}$$

where M is the mass matrix of infinite order, $\dot{y}$ the velocities (complex, zero to peak) of every point on the guitar and $\bar{\dot{y}}$ their complex conjugates. Substitute

$\{\dot{y}\} = [\dot{Y}]\{f\}$ and $\{\bar{\dot{y}}\} = [\bar{\dot{Y}}]\{\bar{f}\}$ (where $\dot{Y}$ is the matrix of velocity mobilities and {f} is the vector of all the external forces and moments) into Equation (1).

$$KE = \frac{1}{2}\{f\}^T[\dot{Y}]^T[M][\bar{\dot{Y}}]\{\bar{f}\} \tag{2}$$

Substitute

$[\dot{Y}] = [\Phi]\left[\frac{Y_i^*}{M_i^*}\right][\Phi]^T$ and $[M] = [\Phi]^{-T}\left[M_i^*\right][\Phi]^{-1}$, (where Φ is the matrix of normal modes, M_i is the generalized mass of the i-th mode and

$$\dot{Y}_i^* = \frac{\omega}{\Omega_i^2}\frac{g_i + i(1-\omega^2/\Omega_i^2)}{(1-\omega^2/\Omega_i^2)^2 + g_i^2}$$

where Ω_i is the i-th natural frequency, ω is the forcing frequency and g is the structural damping). This gives

$$KE = \frac{1}{2}\{f\}^T[\Phi]\left[\frac{\dot{Y}^*\bar{\dot{Y}}^*}{M_i^*}\right][\Phi]^T\{\bar{f}\} \tag{3}$$

Let $\dot{\mathcal{F}}_i \equiv \dot{Y}_i^*\bar{\dot{Y}}_i^* = \frac{\omega^2}{\Omega_i^4}\frac{1}{(1-\frac{\omega^2}{\Omega_i^2})^2 + g_i^2}$, noting that it is independent of the generalized coordinates (i.e., locations on the guitar). The triple matrix product in the center of Equation (3) can be expressed as

$$\sum_{i=1}^{\infty}\left[\frac{\dot{\mathcal{F}}_i}{M_i^*}\right]\{\phi\}_i\{\phi\}_i^T = \sum_{i=1}^{\infty}\left[\dot{\mathcal{F}}_i\right][A]_i$$ where A_i is the matrix of modal accelerations for the i-th mode and the diagonal terms of $\left[\dot{\mathcal{F}}_i\right]$ are identical. Substituting, we can write Equation (3) as

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$$K\mathcal{E} = \frac{1}{2}\sum_{i=1}^{\infty} \{f\}^T [A]_i \{F\} \dot{\mathcal{F}}_i$$ where $\dot{\mathcal{F}}_i$ is a scalar. (4)

Up to this point the matrices have been carried as infinitely large to include every point on the guitar, but we see from Equation (4) that every term in the matrix of modal accelerations relating to points at which there are no external forces or moments (i.e., points other than the generalized coordinates at the bridge) are zero. This leads to the interesting conclusion that the kinetic energy of the entire guitar structure at any frequency is a function of only those modal accelerations at and among the coordinates of the exciting forces and moments at the bridge. The same conclusion holds for the potential energy.

This does not imply that the dynamics properties only at the bridge determine the total vibratory energy but, rather, says that the total energy can be measured by data taken only at the bridge.

Examining the expression for $\dot{\mathcal{F}}_i$ in terms of the forcing frequency, natural frequencies and damping coefficient, we see that it drops off rapidly in magnitude on either side of the natural frequency. This tells us that the kinetic or potential energy at any frequency is negligibly influenced by natural frequencies far below or above that frequency. This is in contrast to the real acceleration or displacement (or imaginary velocity) mobility which is significantly affected at any frequency by all the lower modes as well as the proximate modes. Instead of summing Equation (4) from 1 to infinity, we can sum it over only the proximate modes for all practical purposes.

The vector of forces and moments externally applied can presumably be considered a function of how the instrument is played, being forces and moments from the strings, as opposed to properties of the guitar itself. This leaves us with (1) modal accelerations at the bridge, (2) natural frequencies and (3) the damping coefficients which define the properties of the instrument in terms of the vibratory energy at any frequency. It is further seen from Equation (4) that the energy produced by any given mode at its natural frequency and for a given excitation is a function only of the modal accelerations at the bridge for that mode and the damping. The technology for measuring these is better established at Kaman than elsewhere.

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If we can assume that the only excitation of significance is a shear at the bridge, Equation (4) reduces to

$$K\mathcal{E} \cong \frac{1}{2}\sum f^2 A_{bbi} \dot{\mathcal{F}}_i$$

where the summation is over only the proximate modes.

The kinetic energy for any mode i is

$$K\mathcal{E}_i = \frac{f^2}{2\omega^2} A_{bbi} \frac{\omega^4/\Omega_i^4}{(1-\omega^2/\Omega_i^2)^2 + g_i^2} \quad (5)$$

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In either case the driving point mobility at the bridge is the only measurement required to determine the kinetic (and potential) energy of the guitar in the frequency range of interest. From this measurement, over the range of frequencies, the modal accelerations at the driving point and the damping can be determined.

From the Antiresonance Theory paper of September 1971, the i-th modal acceleration at the bridge is given by

$$A_{bbi} = C_{bb}\,\Omega_i^2 \frac{\prod_{j=1}^{N}\left(1 - \Omega_i^2/a_j^2\right)}{\prod_{\substack{j=1 \\ j\neq i}}^{N}\left(1 - \Omega_i^2/\Omega_j^2\right)} \qquad (6)$$

where C_{bb} is the influence coefficient (the deflection at the bridge per unit force at the bridge) and the a's are the antiresonant frequencies at the bridge. Because this is a driving point, there will be one antiresonance between each pair of natural frequencies. From Equations (6) and (5) it is seen that:

(1) The closer a natural frequency is to an antiresonance frequency the smaller the kinetic energy of that mode.

(2) The greater the influence coefficient at the bridge (i.e., the greater the flexibility of the top) the greater the kinetic energy of every mode. This seems to explain why "purfling" of a violin (Ref. Backus, J., "The Acoustical Foundations of Music", W. W. Norton & Co., New York, 1969, p. 178) has such a great effect on its tone and power.

Many types of changes can be made in a guitar structure to change the kinetic energy of the modes but the easiest to predict and try are mass changes. For example, from WP 74-13 we see that the addition of mass at the bridge changes the acceleration mobility by

$$\ddot{Y}'_{bb} = \frac{\ddot{Y}_{bb}}{1 + \Delta m_{bb}\,\ddot{Y}_{bb}} \qquad (7)$$

and lowers the natural frequencies while leaving the antiresonances unchanged. In some instances, which can be predicted from an examination of the undamped driving point acceleration mobility curve, the addition of this mass would give a greater separation of some natural frequencies and antiresonances and, therefore as seen from Equation (6), increase the kinetic energy of these modes. The larger the kinetic energy of each mode (i.e., the greater the modal acceleration) the less critical is the tuning of the natural frequencies; tuning of the structure and moving the natural frequencies to increase the modal energies is, therefore, a matter of trade-offs. Adding mass at other than the bridge, it appears from a recent finding which will be discussed in a forthcoming Working Paper, moves both the natural frequencies and the bridge antiresonances, by amounts that can be predicted using only mobility test data, thereby changing the kinetic energy of the modes. Other methods of using mobility test data, which would apply to guitar structures, are covered in numerous Working Papers and reports.

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main function of the varnish appears to be to seal the pores of the wood to keep out dust, which otherwise would adversely affect the properties of the wood. The effect of varnish is still being studied.[12]

The question of aging is being looked into; it does not appear that an old violin is better than a new one simply because it is older. Some changes of the properties of wood with age appear to improve violin tone; these are being investigated. However, age is not the only factor.

The various properties of woods used in violin making have been measured.[13] The values of density, elasticity, and damping that appear to be best have been worked out, but other properties still remain to be evaluated. If the various properties of woods that are important to violin tone can be isolated and measured, it should be possible to develop synthetic materials with these properties from which to manufacture violins. (Whether string players would accept instruments made of synthetic materials is another question.)

It does appear that playing a new violin can improve its tone. One reason is as follows: the vibrations of the top plate of the violin are affected by the fact that it is glued firmly to the ribs, and it has been found that the tone of the violin is better if the joint between top plate and ribs is relatively flexible. As an example, it was found that a cheap violin could have its tone improved and power increased considerably by cutting a narrow groove around the outside edge of the top plate; this would allow for more bending at this joint.[14] The so-called *purfling* of the violin is just such a narrow groove cut around the outside edge of the top plate and filled with thin strips of wood that are glued in, it is shown in Fig. 4(a). When new, the joint between the top and ribs is relatively stiff, but as the violin is played the glue holding the strips of wood in the groove apparently cracks loose to some extent and allows the joint more flexibility. After a period of playing, the purfling becomes sufficiently loosened (but not so loose as to fall out) and the tone of the violin is better.

There are other reasons why the tone of a violin is improved by playing it. These have to do with such things as changes in the wood structure that occur when the wood is stretched and compressed by continual vibrations. These matters still need much investigation.

The vibrato, such a characteristic part of violin playing, has been investigated and found to be a fluctuation in both amplitude and frequency. The amplitude variation is not as important as the frequency variation, which is what gives the tone its characteristic quality. This frequency variation is to be expected since it is produced by moving the finger back and forth on the string and so periodically varying its length. The frequency of the vibrato (as distinguished from the frequency of the string) is some five to six vibrations per second; the frequency of the string varies over a range of some 30 to 45 cents.[15]

strings will then produce smaller accelerations and hence smaller amplitude vibrations of the bridge. As a consequence, there will be a reduction in amplitude of the harmonics in the string tone, and the amount of reduction is greater the higher the frequencies of the harmonics. Hence the tone of the instrument with the mute in place is softer and of a less bright quality than without the mute.

The Violin Body

The hollow box that forms the body of the violin serves two functions: it supports the strings so they can vibrate properly, and it makes the sound from the strings audible. The vibrating strings themselves disturb very little air and hence radiate practically no sound; the important job of the violin body is to transmit these string vibrations to the air in such a way as to give them the proper loudness and tone. The quality of the sound the violin radiates will depend not only on the harmonic structure of the vibration of the string, but also on the way in which these harmonics are transmitted through the bridge to the body of the violin and thence to the air.

The essential parts of the body of the violin are shown in Figs. 1 and 4. Fig. 4(a) is a lateral section taken at the point marked *A* in Fig. 1, and Fig. 4(b) is a longitudinal section through the center.

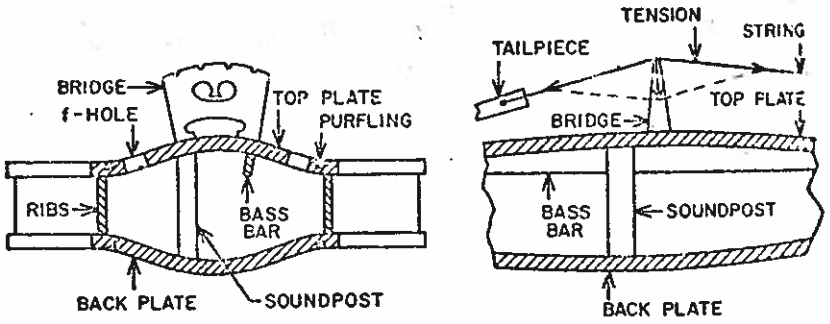


FIG. 4. (a) Lateral section of a violin, taken at the point marked *A* in Fig. 1. The plate thicknesses are much exaggerated. (b) Longitudinal section, showing forces exerted by strings on the bridge.

The violin box, of elaborate shape, consists of two thin plates of wood: the *back plate*, the side away from the strings; and the *top plate* or *belly*, the side adjacent to the strings. These plates are glued to strips of wood called *ribs*, which form the sides of the box and give the instrument its characteristic shape. The air space inside the box communicates to the outside through the *f-holes* cut in the top plate. The bridge carrying the four strings is mounted on the top plate midway between the f-holes. The (unbowed) strings passing over the bridge